The Open University

Faculty of Science, Technology, Engineering and Mathematics
M208 Pure Mathematics

Additional exercises for Book F

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2.1

Additional exercises for Unit F1

Section 1

Additional Exercise F1

Determine whether each of the following limits exists, and evaluate those limits which do exist.

(a) $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$ (b) $\lim_{x\to 1} \dfrac{x^3-1}{|x-1|}$ (c) $\lim_{x\to 2} e^{x^2}$

(d) $\lim_{x\to 0^+} \dfrac{\cos(1/x^2)}{x}$

Hint: In part (a), use the identity $x^3-1=(x-1)(x^2+x+1)$.

Additional Exercise F2

Determine the following limits.

(a) $\lim_{x\to 0} \left(\sin x + \dfrac{e^x-1}{x}\right)$ (b) $\lim_{x\to 0} \dfrac{e^{|x|}-1}{|x|}$

(c) $\lim_{x\to 1^-} \dfrac{x^3-1}{|x-1|}$

Section 2

Additional Exercise F3

Prove that:

(a) $\dfrac{1}{x^4} \to \infty$ as $x \to 0$

(b) $\cot x \to \infty$ as $x \to 0^+$

(c) $e^x - x \to \infty$ as $x \to \infty$

(d) $\log x \to -\infty$ as $x \to 0^+$

(e) $x + \sin x \to \infty$ as $x \to \infty$

(f) $x^x \to \infty$ as $x \to \infty$

(g) $x^{1/x} \to 1$ as $x \to \infty$

(h) $\dfrac{x^2+\log x}{x+e^x} \to 0$ as $x \to \infty$.

Hint: In parts (d), (f) and (g), use the identities

$$\log x = -\log(1/x) \quad \text{and} \quad a^x = \exp(x \log a).$$

In part (h), use the solution to Exercise F9(b) in Unit F1.

Section 3

Additional Exercise F4

Use the ε-δ definition to prove that each of the following functions f is continuous at the given point c:

(a) $f(x) = 6x^2 - x, \quad c = -1$

(b) $f(x) = x^5, \quad c = 0$

(c) $f(x) = \sqrt{x}, \quad c = 4$

(d) $f(x) = \dfrac{1}{x}, \quad c = 1.$

Hint: In part (c), use the fact that $(\sqrt{x}-2)(\sqrt{x}+2) = x-4$.

Additional Exercise F5

Use the ε-δ definition to evaluate

$$\lim_{x\to -1} \frac{x^3+1}{x+1}.$$

Section 4

Additional Exercise F6

Use Strategy F4 to determine whether each of the following functions is uniformly continuous on the given interval:

(a) $f(x) = x, \quad I = \mathbb{R}$

(b) $f(x) = \dfrac{1}{x^2}, \quad I = (0,3]$

(c) $f(x) = \dfrac{2x}{1+x}, \quad I = [0,\infty)$

(d) $f(x) = e^x, \quad I = \mathbb{R}.$

Hint: In part (d), you may find it helpful to use the inequality $e^x \geq 1 + x \geq x$, for $x \geq 0$.

Additional Exercise F7

Use Theorem F19 to prove that the function in Additional Exercise F6(c) is uniformly continuous on the interval $[-\frac{1}{2}, 0]$.

Solutions to additional exercises for Unit F1

Solution to Additional Exercise F1

(a) The domain of f is $\mathbb{R} - \{1\}$, so f is defined on each punctured neighbourhood of 1. Also,

$$f(x) = \frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{x-1}$$
$$= x^2 + x + 1, \quad \text{for } x \neq 1.$$

Thus if (x_n) is any sequence in $\mathbb{R} - \{1\}$ such that $x_n \to 1$, then

$$f(x_n) = x_n^2 + x_n + 1 \to 1 + 1 + 1 = 3 \ \text{ as } n \to \infty,$$

by the Combination Rules for sequences. Hence

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = 3.$$

(b) The presence of the modulus in the denominator of

$$f(x) = \frac{x^3-1}{|x-1|}$$

suggests that the limit of $f(x)$ as x tends to 1 does not exist, because the quotient may behave in a different way on either side of 1.

Therefore, we use the first part of Strategy F1 and consider the two sequences

$$x_n = 1 + 1/n, \quad y_n = 1 - 1/n, \quad n = 1, 2, \dots.$$

The function is defined on $\mathbb{R} - \{1\}$ and both sequences tend to 1 from within $\mathbb{R} - \{1\}$, but

$$\begin{aligned} f(x_n) &= \frac{(1+1/n)^3 - 1}{|(1+1/n) - 1|} \\ &= \frac{(1 + 3/n + 3/n^2 + 1/n^3) - 1}{1/n} \\ &= 3 + 3/n + 1/n^2 \to 3, \text{ as } n \to \infty, \end{aligned}$$

whereas

$$\begin{aligned} f(y_n) &= \frac{(1-1/n)^3 - 1}{|(1-1/n) - 1|} \\ &= \frac{(1 - 3/n + 3/n^2 - 1/n^3) - 1}{1/n} \\ &= -3 + 3/n - 1/n^2 \to -3, \text{ as } n \to \infty. \end{aligned}$$

Hence

$$\lim_{x \to 1} \frac{x^3-1}{|x-1|} \text{ does not exist.}$$

(c) The function $x \longmapsto e^{x^2}$ is defined and continuous on $\mathbb{R}$. Hence, by Theorem F2,

$$\lim_{x \to 2} e^{x^2} = e^4.$$

(d) The term $1/x^2$ is large near 0, so $\cos(1/x^2)$ is highly oscillatory for x near 0. This suggests that this one-sided limit does not exist. Therefore, we use the one-sided limit version of the second part of Strategy F1.

Consider the sequence

$$x_n = \frac{1}{\sqrt{2n\pi}}, \quad n = 1, 2, \dots,$$

chosen because

$$\cos(1/x_n^2) = \cos(2n\pi) = 1, \quad \text{for } n = 1, 2, \dots.$$

Then $x_n > 0$ and $x_n \to 0$, and

$$\begin{aligned} f(x_n) &= \frac{\cos(1/x_n^2)}{x_n} \\ &= \frac{\cos(2n\pi)}{1/\sqrt{2n\pi}} \\ &= \sqrt{2n\pi} \to \infty \ \text{ as } n \to \infty. \end{aligned}$$

Hence

$$\lim_{x \to 0^+} \frac{\cos(1/x^2)}{x} \text{ does not exist.}$$

Solution to Additional Exercise F2

(a) The function $x \longmapsto \sin x$ is defined and continuous on $\mathbb{R}$. Hence, by Theorem F2,

$$\lim_{x \to 0} \sin x = \sin 0 = 0.$$

We also know that

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1;$$

see Theorem F6(c). Hence, by the Sum Rule,

$$\lim_{x \to 0} \left(\sin x + \frac{e^x - 1}{x} \right) = 0 + 1 = 1.$$

(b) We can write

$$\frac{e^{|x|} - 1}{|x|} = g(f(x)),$$

where $f(x) = |x|$ and $g(x) = \dfrac{e^x - 1}{x}$.

Substituting $u = f(x) = |x|$, and using the fact that u is continuous at 0, we have

$$u = |x| \to 0 \ \text{ as } x \to 0$$

and

$$g(u) = \frac{e^u - 1}{u} \to 1 \ \text{ as } u \to 0.$$

The first proviso to the Composition Rule holds because $f(x) = |x| \neq 0$ in $N_1(0)$, for example. Thus, by the Composition Rule,

$$g(f(x)) = \frac{e^{|x|} - 1}{|x|} \to 1 \quad \text{as } x \to 0.$$

(c) First, the function

$$f(x) = \frac{x^3 - 1}{|x - 1|}$$

is defined on $(-\infty, 1)$.

Next, for $x < 1$ we have $|x - 1| = 1 - x$, so

$$f(x) = \frac{x^3 - 1}{1 - x} = -(x^2 + x + 1), \quad \text{for } x < 1.$$

Thus if (x_n) lies in $(-\infty, 1)$ and $x_n \to 1$, then

$$f(x_n) = -(x_n^2 + x_n + 1) \to -(1 + 1 + 1) = -3,$$

by the Combination Rules for sequences. Hence

$$\lim_{x \to 1^-} \frac{x^3 - 1}{|x - 1|} = -3.$$

Solution to Additional Exercise F3

(a) Let $f(x) = x^4$; then $f(x) > 0$ for $x \in \mathbb{R} - \{0\}$, and

$$\lim_{x \to 0} x^4 = 0,$$

since f is continuous at 0.

Hence, by the Reciprocal Rule,

$$\frac{1}{f(x)} = \frac{1}{x^4} \to \infty \quad \text{as } x \to 0.$$

(b) Let

$$f(x) = \frac{1}{\cot x} = \tan x;$$

then

$$f(x) > 0, \quad \text{for } 0 < x < \pi/2,$$

and f is continuous on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ with $f(0) = 0$, so

$$f(x) \to 0 \quad \text{as } x \to 0^+.$$

Hence, by the Reciprocal Rule,

$$\frac{1}{f(x)} = \cot x \to \infty \quad \text{as } x \to 0^+.$$

(c) Let

$$f(x) = \frac{1}{e^x - x};$$

then

$$f(x) > 0, \quad \text{for } x > 0,$$

since $e^x > 1 + x$ for $x > 0$, and

$$f(x) = \frac{1/e^x}{1 - x/e^x} \to \frac{0}{1 - 0} = 0 \quad \text{as } x \to \infty,$$

by Theorem F13(b) and the Combination Rules.

Hence, by the Reciprocal Rule,

$$\frac{1}{f(x)} = e^x - x \to \infty \quad \text{as } x \to \infty.$$

(Alternatively, since $e^x = 1 + x + x^2/2! + \cdots$, for $x > 0$, we have

$$e^x - x > x^2/2, \quad \text{for } x > 0,$$

so

$$e^x - x \to \infty \quad \text{as } x \to \infty,$$

by Theorem F11(a) and part (b) of the Squeeze Rule.)

(d) We can write

$$\log x = -\log(1/x) = g(f(x)),$$

where $f(x) = 1/x$ and $g(x) = -\log x$.

Substituting $u = f(x) = 1/x$, we have

$$u = 1/x \to \infty \quad \text{as } x \to 0^+ \text{ (Reciprocal Rule)},$$
$$g(u) = -\log u \to -\infty \quad \text{as } u \to \infty,$$

by Theorem F13(c) and the Multiple Rule.

Thus, by the Composition Rule,

$$g(f(x)) = \log x \to -\infty \quad \text{as } x \to 0^+.$$

(e) Since

$$-1 \le \sin x \le 1, \quad \text{for } x \in \mathbb{R},$$

we have

$$x + \sin x \ge x - 1, \quad \text{for } x \in \mathbb{R}.$$

By Theorem F11(a), we have

$$x - 1 \to \infty \quad \text{as } x \to \infty,$$

so, by part (b) of the Squeeze Rule,

$$x + \sin x \to \infty \quad \text{as } x \to \infty.$$

(f) First note that for $x > 0$,

$$x^x = \exp(x \log x) = g(f(x)),$$

where $f(x) = x \log x$ and $g(x) = \exp(x)$.

Substituting $u = x \log x$, we have

$$u = x \log x \to \infty \quad \text{as } x \to \infty,$$

by Theorem F13(c) and the Product Rule, and

$$\exp(u) = e^u \to \infty \text{ as } u \to \infty,$$

by Theorem F13(b). Thus, by the Composition Rule,

$$g(f(x)) = x^x \to \infty \text{ as } x \to \infty.$$

(Alternatively, for $x \geq e$ we have

$$x^x \geq e^x,$$

so we can deduce the result from part (b) of the Squeeze Rule and Theorem F13(b).)

(g) First note that for $x > 0$,

$$x^{1/x} = \exp\left(\frac{1}{x}\log x\right) = g(f(x)),$$

where $f(x) = (\log x)/x$ and $g(x) = \exp(x)$.

Substituting $u = (\log x)/x$, we have

$$u = (\log x)/x \to 0 \text{ as } x \to \infty,$$

by Theorem F13(c), and

$$\exp(u) = e^u \to 1 \text{ as } u \to 0,$$

since g is continuous at 0. Thus, by the Composition Rule,

$$g(f(x)) = x^{1/x} \to 1 \text{ as } x \to \infty.$$

(h) We have

$$\frac{x^2 + \log x}{x + e^x} = \frac{x^2/e^x + (\log x)/e^x}{x/e^x + 1}.$$

Thus, by the solution to Exercise F9(b) in Unit F1, Theorem F13(b) and the Combination Rules,

$$\frac{x^2 + \log x}{x + e^x} \to \frac{0+0}{0+1} = 0 \text{ as } x \to \infty.$$

Solution to Additional Exercise F4

(a) The domain of $f(x) = 6x^2 - x$ is $\mathbb{R}$.

Let $\varepsilon > 0$ be given. We want to choose $\delta > 0$, in terms of ε, such that

$$|f(x) - f(-1)| < \varepsilon, \text{ for all } x \text{ with } |x+1| < \delta. \quad (*)$$

We follow the steps in Strategy F3.

1. First we write

$$f(x) - f(-1) = 6x^2 - x - 7 = (x+1)(6x-7).$$

2. Next we obtain an upper bound for $|6x - 7|$ when x is near -1. If $|x+1| \leq 1$, then x lies in the interval $[-2, 0]$, so (by the Triangle Inequality)

$$\begin{aligned}|6x - 7| &\leq |6x| + |-7| = 6|x| + 7 \\ &\leq 6 \times 2 + 7 = 19.\end{aligned}$$

3. Hence

$$|f(x) - f(-1)| \leq 19|x+1|, \quad \text{for } |x+1| \leq 1.$$

So if $|x+1| < \delta$, where $0 < \delta \leq 1$, then

$$|f(x) - f(-1)| < 19\delta.$$

Thus, if we choose $\delta = \min\{1, \frac{1}{19}\varepsilon\}$, then

$$|f(x) - f(-1)| < 19\delta \leq \varepsilon, \quad \text{for all } x \text{ with } |x+1| < \delta,$$

which proves statement $(*)$.

Thus f is continuous at the point -1.

(b) The domain of $f(x) = x^5$ is $\mathbb{R}$.

Let $\varepsilon > 0$ be given. We want to choose $\delta > 0$, in terms of ε, such that

$$|f(x) - f(0)| < \varepsilon, \quad \text{for all } x \text{ with } |x| < \delta. \quad (*)$$

In this case it is possible to use Strategy F3, but it is easier to note that

$$|f(x) - f(0)| < \varepsilon \quad \text{is equivalent to} \quad |x|^5 < \varepsilon.$$

Thus, if we choose $\delta = \sqrt[5]{\varepsilon}$, then

$$|f(x) - f(0)| = |x|^5 < \delta^5 = \varepsilon, \text{ for all } x \text{ with } |x| < \delta,$$

which proves statement $(*)$.

Thus f is continuous at the point 0.

(c) The domain of $f(x) = \sqrt{x}$ is $[0, \infty)$.

Let $\varepsilon > 0$ be given. We want to choose $\delta > 0$, in terms of ε, such that

$$|f(x) - f(4)| < \varepsilon, \text{ for all } x \text{ with } |x-4| < \delta. \quad (*)$$

We follow the steps in Strategy F3.

1. First we use the hint to write

$$f(x) - f(4) = \sqrt{x} - 2 = \frac{x-4}{\sqrt{x}+2}.$$

2. Next we obtain an upper bound for

$$\left|\frac{1}{\sqrt{x}+2}\right|$$

when x is near 4. If $|x-4| \leq 1$, then x lies in the interval $[3,5]$, so $\sqrt{x} + 2 \geq \sqrt{3} + 2$, and thus

$$\left|\frac{1}{\sqrt{x}+2}\right| \leq \frac{1}{\sqrt{3}+2} \leq 1.$$

3. Hence
$$|f(x) - f(4)| \le |x-4|, \quad \text{for } |x-4| \le 1.$$
So if $|x-4| < \delta$, where $0 < \delta \le 1$, then
$$|f(x) - f(4)| < \delta.$$
Thus, if we choose $\delta = \min\{1, \varepsilon\}$, then
$$|f(x) - f(4)| < \delta \le \varepsilon, \quad \text{for all } x \text{ with } |x-4| < \delta,$$
which proves statement $(*)$.

Thus f is continuous at the point 4.

(d) The domain of $f(x) = 1/x$ is $\mathbb{R} - \{0\}$.

Let $\varepsilon > 0$ be given. We want to choose $\delta > 0$, in terms of ε, such that
$$|f(x) - f(1)| < \varepsilon, \quad \text{for all } x \text{ with } |x-1| < \delta. \quad (*)$$

We follow the steps in Strategy F3.

1. First we write
$$f(x) - f(1) = \frac{1}{x} - 1 = \frac{1-x}{x} = (x-1)\left(\frac{-1}{x}\right).$$
2. Next we obtain an upper bound for $|-1/x|$ when x is near 1. If $|x-1| \le \frac{1}{2}$ (chosen to avoid the point 0), then x lies in the interval $[\frac{1}{2}, \frac{3}{2}]$, so $x \ge \frac{1}{2}$ and hence
$$\left|\frac{-1}{x}\right| = \frac{1}{x} \le 2.$$
3. Hence
$$|f(x) - f(1)| \le 2|x-1|, \quad \text{for } |x-1| \le \tfrac{1}{2}.$$
So if $|x-1| < \delta$, where $0 < \delta \le \frac{1}{2}$, then
$$|f(x) - f(1)| < 2\delta.$$
Thus, if we choose $\delta = \min\{\frac{1}{2}, \frac{1}{2}\varepsilon\}$, then
$$|f(x) - f(1)| < 2\delta \le \varepsilon, \quad \text{for all } x \text{ with } |x-1| < \delta,$$
which proves statement $(*)$.

Thus f is continuous at the point 1.

Solution to Additional Exercise F5

The domain of
$$f(x) = \frac{x^3+1}{x+1}$$
is $\mathbb{R} - \{-1\}$, so f is defined on each punctured neighbourhood of -1. Also, for $x \ne -1$,
$$f(x) = \frac{x^3+1}{x+1} = \frac{(x+1)(x^2-x+1)}{x+1} = x^2 - x + 1.$$

This suggests that
$$\lim_{x \to -1} f(x) = (-1)^2 - (-1) + 1 = 3,$$
so we must prove that

for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 3| < \varepsilon$, for all x with $0 < |x+1| < \delta$. $(*)$

1. First we write, for $x \ne -1$,
$$f(x) - 3 = x^2 - x + 1 - 3 = x^2 - x - 2 = (x+1)(x-2).$$
2. Next, if $|x+1| \le 1$, then x lies in the interval $[-2, 0]$, so (by the Triangle Inequality)
$$|x-2| \le |x| + 2 \le 2 + 2 = 4.$$
3. Hence
$$|f(x) - 3| \le 4|x+1|, \quad \text{for } 0 < |x+1| \le 1.$$
So if $0 < |x+1| < \delta$, where $0 < \delta \le 1$, then
$$|f(x) - 3| < 4\delta.$$
Thus if we choose $\delta = \min\{1, \frac{1}{4}\varepsilon\}$, then
$$|f(x) - 3| < 4\delta \le \varepsilon, \quad \text{for all } x \text{ with } 0 < |x+1| < \delta,$$
which proves statement $(*)$.

Hence
$$\lim_{x \to -1} f(x) = 3.$$

Solution to Additional Exercise F6

(a) We use the first part of Strategy F4. Let $\varepsilon > 0$ be given. For $x, y \in \mathbb{R}$, we have
$$f(x) - f(y) = x - y,$$
so
$$|f(x) - f(y)| = |x - y|.$$
Thus, if we choose $\delta = \varepsilon$, then whenever $x, y \in \mathbb{R}$ and $|x - y| < \delta$, we have
$$|f(x) - f(y)| = |x - y| < \delta = \varepsilon.$$
Hence f is uniformly continuous on $\mathbb{R}$.

(b) Following the second part of Strategy F4, we take $x_n = 1/(2n)$ and $y_n = 1/n$, for $n = 1, 2, \ldots$. Both sequences lie in $I = (0, 3]$ and

$$|x_n - y_n| = \left|\frac{1}{2n} - \frac{1}{n}\right| = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

$$\begin{aligned} |f(x_n) - f(y_n)| &= \left|\frac{1}{x_n^2} - \frac{1}{y_n^2}\right| \\ &= \left|\frac{1}{(1/(2n))^2} - \frac{1}{(1/n)^2}\right| \\ &= 4n^2 - n^2 \\ &= 3n^2 \geq 3, \quad \text{for } n = 1, 2, \ldots. \end{aligned}$$

Thus, by taking $\varepsilon = 3$ in the second part of Strategy F4, we deduce that f is not uniformly continuous on $(0, 3]$.

(c) We use the first part of Strategy F4. Let $\varepsilon > 0$ be given. For $x, y \in [0, \infty)$, we have

$$\begin{aligned} f(x) - f(y) &= \frac{2x}{1+x} - \frac{2y}{1+y}, \\ &= \frac{2(x-y)}{(1+x)(1+y)}. \end{aligned}$$

Since $1 + x \geq 1$ and $1 + y \geq 1$, for $x, y \in [0, \infty)$,

$$|f(x) - f(y)| \leq 2|x - y|, \quad \text{for } x, y \in [0, \infty).$$

Thus, if we choose $\delta = \frac{1}{2}\varepsilon$, then whenever $x, y \in [0, \infty)$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq 2|x - y| < 2\delta = \varepsilon.$$

Hence f is uniformly continuous on $[0, \infty)$.

(d) Following the second part of Strategy F4, we take $x_n = n + 1/n$ and $y_n = n$, for $n = 1, 2, \ldots$. Both sequences lie in $I = \mathbb{R}$ and

$$|x_n - y_n| = (n + 1/n) - n = 1/n \to 0 \quad \text{as } n \to \infty,$$

$$\begin{aligned} |f(x_n) - f(y_n)| &= |e^{x_n} - e^{y_n}| \\ &= e^{n+1/n} - e^n \\ &= e^n(e^{1/n} - 1) \\ &\geq n \times (1/n) = 1, \quad \text{for } n = 1, 2, \ldots, \end{aligned}$$

by using the inequality $e^x \geq 1 + x \geq x$, for $x \geq 0$.

Thus, by taking $\varepsilon = 1$ in the second part of Strategy F4, we deduce that f is not uniformly continuous on $\mathbb{R}$.

Solution to Additional Exercise F7

Since $f(x) = 2x/(1 + x)$ is a rational function, it is continuous on its domain $\mathbb{R} - \{-1\}$ and hence on the bounded closed interval $[-\frac{1}{2}, 0]$. We deduce that f is uniformly continuous on $[-\frac{1}{2}, 0]$, by Theorem F19.

Additional exercises for Unit F2

Section 1

Additional Exercise F8

Determine, from the definition of differentiable, which of the following functions f is differentiable at 0. If f is differentiable at 0, then evaluate the derivative $f'(0)$.

(a) $f(x) = \begin{cases} x\sin(1/x^2), & x \neq 0, \\ 0, & x = 0. \end{cases}$

(b) $f(x) = \dfrac{x}{1+x}$

Additional Exercise F9

Use the Glue Rule or Corollary F25 to determine which of the following functions f is differentiable at the given point c. If f is differentiable at c, then evaluate the derivative $f'(c)$.

(a) $f(x) = \begin{cases} -x^2, & x \leq 0, \\ x^3, & x > 0, \end{cases} \quad c = 0.$

(b) $f(x) = \begin{cases} x, & x < 1, \\ x^2, & x \geq 1, \end{cases} \quad c = 1.$

(c) $f(x) = \begin{cases} x, & x < 1, \\ x - x^2, & x \geq 1, \end{cases} \quad c = 1.$

Additional Exercise F10

Prove that the function

$$f(x) = \begin{cases} 1, & x \leq 0, \\ \cos x, & x > 0, \end{cases}$$

is differentiable, and determine the rule of f'.

Section 2

Additional Exercise F11

Verify that the following function is differentiable on $(-1, \infty)$ and determine its derivative, stating which rules you use:

$$f(x) = \log(1+x) + e^{x^2}.$$

Additional Exercise F12

Find (without justification) the derivatives of the following functions.

(a) $f(x) = \dfrac{x^2+1}{x-1} \quad (x \in (1, \infty))$

(b) $f(x) = \log(\sin x) \quad (x \in (0, \pi))$

(c) $f(x) = \log(\sec x + \tan x) \quad (x \in (-\frac{1}{2}\pi, \frac{1}{2}\pi))$

(d) $f(x) = \coth x \quad (x \in \mathbb{R} - \{0\})$

Additional Exercise F13

Prove that the function

$$f(x) = \tanh x \quad (x \in \mathbb{R})$$

has an inverse function f^{-1} that is differentiable on $(-1, 1)$, and find a formula for $(f^{-1})'(x)$.

Additional Exercise F14

Prove that the function

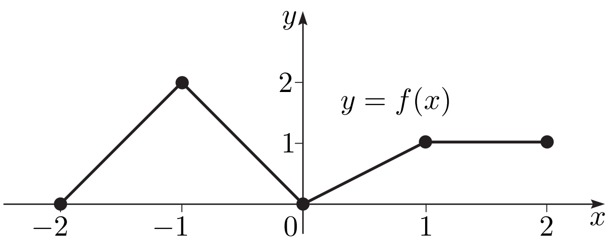
$$f(x) = \tan x + 3x \quad \left(x \in \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right)\right)$$

has an inverse function f^{-1} that is differentiable on $\mathbb{R}$, and find the value of $(f^{-1})'(0)$.

Section 3

Additional Exercise F15

The function f has domain $[-2, 2]$, and its graph consists of four line segments, as shown below.



Identify any

(a) local minima (b) minima

(c) local maxima (d) maxima

of f, and state where these occur.

Additional Exercise F16

Use Rolle's Theorem to show that if

$$f(x) = x^5 - 3x^4 + 2x^3 + 2x^2 - 6x + 1,$$

then there is a value of c in $(1, 2)$ such that $f'(c) = 0$.

Additional Exercise F17

Use Rolle's Theorem to prove that if p is a polynomial and $x_1, x_2, \ldots, x_n$ are distinct zeros of p, then p' has at least $n - 1$ zeros.

Additional Exercise F18

Use Rolle's Theorem to prove that for any real number λ, the function

$$f(x) = x^3 - \tfrac{3}{2}x^2 + \lambda \quad (x \in \mathbb{R}),$$

does not have two distinct zeros in $[0, 1]$.

Hint: Assume that f has two distinct zeros in $[0, 1]$, and deduce a contradiction.

Section 4

Additional Exercise F19

Use the Mean Value Theorem to show that if

$$f(x) = x^3 + 2x^2 + x,$$

then there is a point $c \in (0, 1)$ such that $f'(c) = 4$.

Additional Exercise F20

Let f be continuous on $[0, 2]$ and differentiable on $(0, 2)$, with

$$f(0) = 10 \quad \text{and} \quad |f'(x)| \leq 3, \quad \text{for } x \in (0, 2).$$

Use the Mean Value Theorem to deduce that

$$4 \leq f(2) \leq 16.$$

Additional Exercise F21

Consider the function

$$f(x) = x^3 - 2x^2 + x \quad (x \in \mathbb{R}).$$

(a) Determine those points c such that $f'(c) = 0$.

(b) Using the Second Derivative Test, determine whether the points c found in part (a) correspond to local maxima or local minima, and find the values of these local maxima or local minima.

Additional Exercise F22

Prove the following inequalities.

(a) $\log x \geq 1 - \dfrac{1}{x}, \quad \text{for } x \in [1, \infty)$

(b) $4x^{1/4} \leq x + 3, \quad \text{for } x \in [0, 1]$

(c) $\log(1 + x) > x - \tfrac{1}{2}x^2, \quad \text{for } x \in (0, \infty)$

Section 5

Additional Exercise F23

Use l'Hôpital's Rule to prove that the following limits exist, and evaluate them.

(a) $\displaystyle\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

(b) $\displaystyle\lim_{x \to 1} \frac{(5x + 3)^{1/3} - (x + 3)^{1/2}}{x - 1}$

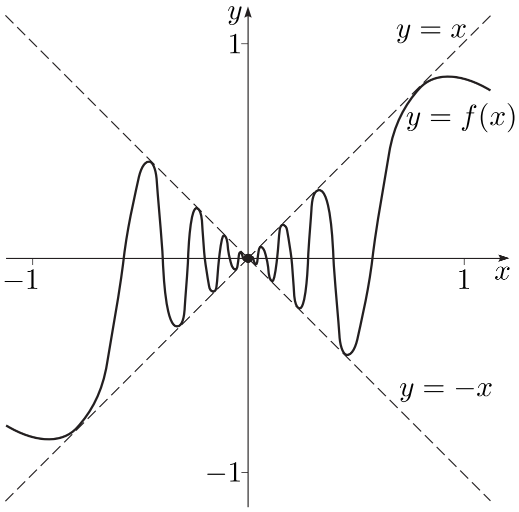
(c) $\displaystyle\lim_{x \to 0} \frac{\sinh x - x}{\sin(x^2)}$

(d) $\displaystyle\lim_{x \to 0} \frac{\sinh(x + \sin x)}{\sin x}$

Solutions to additional exercises for Unit F2

Solution to Additional Exercise F8

(a) The graph of f is shown below. (This is included to aid your understanding – you do not need to sketch the graph as part of your solution.)



We guess that f is not differentiable at 0.

The difference quotient for f at 0 is

$$\begin{aligned} Q(h) &= \frac{f(h) - f(0)}{h} \\ &= \frac{h\sin(1/h^2) - 0}{h} \\ &= \sin(1/h^2), \quad \text{where } h \neq 0. \end{aligned}$$

First, let (h_n) be the null sequence

$$h_n = 1/\sqrt{2n\pi + \tfrac{1}{2}\pi}, \quad n = 0, 1, 2, \ldots,$$

with non-zero terms. Then

$$\begin{aligned} Q(h_n) &= \sin(1/h_n^2) \\ &= \sin(2n\pi + \tfrac{1}{2}\pi) = 1, \quad \text{for } n = 0, 1, 2, \ldots, \end{aligned}$$

so

$$Q(h_n) \to 1 \quad \text{as } n \to \infty.$$

Next, let (k_n) be the null sequence

$$k_n = 1/\sqrt{2n\pi + \tfrac{3}{2}\pi}, \quad n = 0, 1, 2, \ldots,$$

with non-zero terms. Then

$$\begin{aligned} Q(k_n) &= \sin(1/{k_n}^2) \\ &= \sin(2n\pi + \tfrac{3}{2}\pi) = -1, \quad \text{for } n = 0, 1, 2, \ldots, \end{aligned}$$

so

$$Q(k_n) \to -1 \quad \text{as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} Q(h_n) = 1 \neq -1 = \lim_{n\to\infty} Q(k_n).$$

Hence f is not differentiable at 0.

(b) The difference quotient for f at 0 is

$$\begin{aligned} Q(h) &= \frac{f(h) - f(0)}{h} \\ &= \frac{h/(1+h) - 0}{h} \\ &= \frac{1}{1+h}, \quad \text{where } h \neq 0. \end{aligned}$$

Thus $Q(h) \to 1$ as $h \to 0$. Hence f is differentiable at 0, with $f'(0) = 1$.

Solution to Additional Exercise F9

(a) Let $I = \mathbb{R}$ and define

$$g(x) = -x^2 \quad (x \in \mathbb{R}) \quad \text{and} \quad h(x) = x^3 \quad (x \in \mathbb{R}).$$

Then

$$\begin{aligned} f(x) &= g(x), \quad \text{for } x < 0, \\ f(x) &= h(x), \quad \text{for } x > 0, \end{aligned}$$

so condition 1 of the Glue Rule holds (with $c = 0$).

Furthermore, $f(0) = g(0) = h(0) = 0$, so condition 2 holds.

Also, g and h are differentiable with

$$g'(x) = -2x \quad (x \in \mathbb{R}) \quad \text{and} \quad h'(x) = 3x^2 \quad (x \in \mathbb{R}),$$

so condition 3 holds.

Since $g'(0) = h'(0) = 0$, it follows from the Glue Rule that f is differentiable at 0 and $f'(0) = 0$.

(b) Let $I = \mathbb{R}$ and define

$$g(x) = x \quad (x \in \mathbb{R}) \quad \text{and} \quad h(x) = x^2 \quad (x \in \mathbb{R}).$$

Then

$$\begin{aligned} f(x) &= g(x), \quad \text{for } x < 1, \\ f(x) &= h(x), \quad \text{for } x > 1, \end{aligned}$$

so condition 1 of the Glue Rule holds (with $c = 1$).

Furthermore, $f(1) = g(1) = h(1) = 1$, so condition 2 holds.

Also, g and h are differentiable with

$$g'(x) = 1 \quad (x \in \mathbb{R}) \quad \text{and} \quad h'(x) = 2x \quad (x \in \mathbb{R}),$$

so condition 3 holds.

Since $g'(1) = 1$ and $h'(1) = 2$, we have $g'(1) \neq h'(1)$. Thus, by the Glue Rule, f is not differentiable at 1.

(c) The function f is not continuous at the point 1, since

$$f(1) = 0, \quad \text{because } f(x) = x - x^2 \text{ for } x \geq 1,$$

but

$$\lim_{x \to 1^-} f(x) = 1, \quad \text{because } f(x) = x \text{ for } x < 1.$$

Thus f is not differentiable at 1, by Corollary F25.

Solution to Additional Exercise F10

First, let $I = \mathbb{R}$ and define

$$g(x) = 1 \quad (x \in \mathbb{R})$$

and

$$h(x) = \cos x \quad (x \in \mathbb{R}).$$

Then

$$\begin{aligned} f(x) &= g(x), \quad \text{for } x < 0, \\ f(x) &= h(x), \quad \text{for } x > 0, \end{aligned} \qquad (*)$$

so condition 1 of the Glue Rule holds (with $c = 0$).

Furthermore, $f(0) = g(0) = h(0) = 1$, so condition 2 holds.

Also, g and h are differentiable with

$$g'(x) = 0 \quad (x \in \mathbb{R})$$

and

$$h'(x) = -\sin x \quad (x \in \mathbb{R}),$$

so condition 3 holds.

Since $g'(0) = h'(0) = 0$, it follows from the Glue Rule that f is differentiable at 0 and $f'(0) = 0$.

Also, by statements $(*)$,

$$\begin{aligned} f'(x) &= g'(x) = 0, \quad \text{for } x < 0, \\ f'(x) &= h'(x) = -\sin x, \quad \text{for } x > 0, \end{aligned}$$

since differentiability is a local property.

Hence f is differentiable, and

$$f'(x) = \begin{cases} 0, & x \leq 0, \\ -\sin x, & x > 0. \end{cases}$$

Solution to Additional Exercise F11

The functions $x \longmapsto 1 + x$ and $x \longmapsto x^2$ are differentiable on $\mathbb{R}$, with derivatives $x \longmapsto 1$ and $x \longmapsto 2x$, respectively.

Since log is differentiable on $\mathbb{R}^+$ and $1 + x$ lies in $\mathbb{R}^+$ whenever $x \in (-1, \infty)$, the function $x \longmapsto \log(1 + x)$ is differentiable on $(-1, \infty)$, with derivative $x \longmapsto 1/(1 + x)$, by the Composition Rule.

Since exp is differentiable on $\mathbb{R}$, the function $x \longmapsto e^{x^2}$ is differentiable on $\mathbb{R}$, with derivative $x \longmapsto 2xe^{x^2}$, by the Composition Rule.

It follows by the Sum Rule that f is differentiable on $(-1, \infty)$, with derivative

$$f'(x) = \frac{1}{x + 1} + 2xe^{x^2}.$$

Solution to Additional Exercise F12

(a)

$$\begin{aligned} f'(x) &= \frac{(x-1)2x - (x^2+1)1}{(x-1)^2} \\ &= \frac{x^2 - 2x - 1}{(x-1)^2} \quad (x \in (1, \infty)) \end{aligned}$$

(b)

$$f'(x) = \frac{1}{\sin x} \cos x = \cot x \quad (x \in (0, \pi))$$

(c)

$$\begin{aligned} f'(x) &= \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x) \\ &= \sec x \quad \left(x \in \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right)\right) \end{aligned}$$

(d) $f(x) = \coth x = \cosh x / \sinh x$, so

$$\begin{aligned} f'(x) &= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} \\ &= \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x \quad (x \in \mathbb{R} - \{0\}). \end{aligned}$$

Solution to Additional Exercise F13

The function

$$f(x) = \tanh x \quad (x \in \mathbb{R})$$

is continuous and strictly increasing, and $f(\mathbb{R}) = (-1, 1)$.

Also, f is differentiable on $\mathbb{R}$, and

$$f'(x) = \operatorname{sech}^2 x \neq 0, \quad \text{for } x \in \mathbb{R}.$$

Thus f satisfies the conditions of the Inverse Function Rule. Hence f has an inverse function f^{-1} that is differentiable on $(-1, 1)$.

If $y = f(x) = \tanh x$, then

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{\operatorname{sech}^2 x}.$$

Now, $\cosh^2 x = 1 + \sinh^2 x$, so $1 = \operatorname{sech}^2 x + \tanh^2 x$, and hence

$$(f^{-1})'(y) = \frac{1}{1 - \tanh^2 x} = \frac{1}{1 - y^2}.$$

Replacing the domain variable y by x, we obtain

$$(\tanh^{-1})'(x) = \frac{1}{1 - x^2} \quad (x \in (-1, 1)).$$

Solution to Additional Exercise F14

Since both of the functions

$$x \longmapsto \tan x \quad \text{and} \quad x \longmapsto 3x$$

are continuous and strictly increasing on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, so is their sum f, and $f((-\frac{1}{2}\pi, \frac{1}{2}\pi)) = \mathbb{R}$.

Also, f is differentiable on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and

$$f'(x) = \sec^2 x + 3 \neq 0, \quad \text{for } x \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi).$$

Thus f satisfies the conditions of the Inverse Function Rule. Hence f has an inverse function f^{-1} that is differentiable on $\mathbb{R}$.

Now, $f(0) = 0$. Hence, by the Inverse Function Rule,

$$(f^{-1})'(0) = \frac{1}{f'(0)} = \frac{1}{\sec^2 0 + 3} = \frac{1}{4}.$$

Solution to Additional Exercise F15

(a) 0, occurring at 0;
1, occurring at all points in $(1, 2)$.

(Note that local minima do not occur at -2 and 2 because no open interval containing either -2 or 2 lies in the domain of f.)

(b) 0, occurring at -2 and 0.

(Note that a function cannot have more than one minimum, but the minimum may occur at more than one point.)

(c) 2, occurring at -1;
1, occurring at all points in $[1, 2)$.

(d) 2, occurring at -1.

Solution to Additional Exercise F16

Since f is a polynomial function, f is continuous on $[1, 2]$ and differentiable on $(1, 2)$. Also, $f(1) = -3$ and $f(2) = -3$, so $f(1) = f(2)$.

Thus f satisfies the conditions of Rolle's Theorem on $[1, 2]$, so there exists a point $c \in (1, 2)$ such that $f'(c) = 0$.

Solution to Additional Exercise F17

We can assume that

$$x_1 < x_2 < \cdots < x_{n-1} < x_n.$$

Since p is a polynomial function, p is continuous on each closed interval $[x_i, x_{i+1}]$ and differentiable on each (x_i, x_{i+1}), for $i = 1, 2, \ldots, n - 1$. Also, $p(x_i) = 0 = p(x_{i+1})$.

Thus p satisfies the conditions of Rolle's Theorem on each interval $[x_i, x_{i+1}]$.

It follows by Rolle's Theorem that p' vanishes at some point in each interval (x_i, x_{i+1}). Since there are $n - 1$ such intervals, p' must have at least $n - 1$ zeros.

Solution to Additional Exercise F18

We suppose that f has two zeros, a and b in $[0, 1]$, with $a < b$, and deduce a contradiction.

Since f is a polynomial function, f is continuous on $[a, b]$ and differentiable on (a, b). Also, we are assuming that $f(a) = f(b) = 0$.

Thus f satisfies the conditions of Rolle's Theorem on $[a, b]$, so there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Now,

$$f'(c) = 3c^2 - 3c = 3c(c - 1).$$

Since $c \in (0, 1)$, it follows that $3c(c - 1)$ cannot be zero. This is the required contradiction.

Solution to Additional Exercise F19

Since $f(x) = x^3 + 2x^2 + x$ is a polynomial function, f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Thus f satisfies the conditions of the Mean Value Theorem on $[0, 1]$. Now

$$\frac{f(1) - f(0)}{1 - 0} = \frac{4 - 0}{1} = 4. \qquad (*)$$

Thus, by the Mean Value Theorem, there exists a point c in $(0, 1)$ such that $f'(c) = 4$.

(Since

$$f'(x) = 3x^2 + 4x + 1,$$

equation $(*)$ can be written as

$$f'(c) = 3c^2 + 4c + 1 = 4.$$

Thus $c = \frac{1}{3}(-2 \pm \sqrt{13})$, so $c = \frac{1}{3}(\sqrt{13} - 2) \approx 0.54$.)

Solution to Additional Exercise F20

The function f satisfies the conditions of the Mean Value Theorem on $[0, 2]$. Hence there exists a point $c \in (0, 2)$ such that

$$\frac{f(2) - f(0)}{2 - 0} = f'(c).$$

Since $|f'(c)| \leq 3$ and $f(0) = 10$, we have

$$|f(2) - 10| = 2|f'(c)| \leq 6.$$

Hence

$$-6 \leq f(2) - 10 \leq 6, \quad \text{so} \quad 4 \leq f(2) \leq 16,$$

as required.

Solution to Additional Exercise F21

(a) We have

$$\begin{aligned} f'(x) &= 3x^2 - 4x + 1 \\ &= (3x - 1)(x - 1). \end{aligned}$$

Thus $f'(x) = 0$ for $x = \frac{1}{3}$ and 1, so the required values of c are $\frac{1}{3}$ and 1.

(b) We have

$$f''(x) = 6x - 4,$$

so

$$f''(\tfrac{1}{3}) = -2 < 0 \quad \text{and} \quad f''(1) = 2 > 0.$$

Also, $f(\frac{1}{3}) = \frac{4}{27}$ and $f(1) = 0$.

Since f is a twice differentiable function defined on $\mathbb{R}$ and f'' is continuous on $\mathbb{R}$, it follows from the Second Derivative Test that f has a local maximum of $\frac{4}{27}$ at $\frac{1}{3}$ and a local minimum of 0 at 1.

Solution to Additional Exercise F22

In each case we follow the steps in Strategy F7.

(a) 1. Let

$$f(x) = \log x - \left(1 - \frac{1}{x}\right) \quad (x \in [1, \infty)).$$

Then f is continuous on $[1, \infty)$ and differentiable on $(1, \infty)$.

2. We have

$$f'(x) = \frac{1}{x} - \frac{1}{x^2} = \frac{x - 1}{x^2} \geq 0, \quad \text{for } x \in (1, \infty),$$

and $f(1) = \log 1 - 0 = 0$.

Thus f is increasing on $[1, \infty)$, by the Increasing–Decreasing Theorem, so

$$f(x) \geq f(1) = 0, \quad \text{for } x \in [1, \infty).$$

Hence

$$\log x \geq 1 - \frac{1}{x}, \quad \text{for } x \in [1, \infty).$$

(b) 1. Let

$$f(x) = x + 3 - 4x^{1/4} \quad (x \in [0, 1]).$$

Then f is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

2. We have

$$f'(x) = 1 - x^{-3/4} \leq 0, \quad \text{for } x \in (0, 1),$$

and $f(1) = 1 + 3 - 4 = 0$.

Thus f is decreasing on $[0, 1]$, by the Increasing–Decreasing Theorem, so

$$f(x) \geq f(1) = 0, \quad \text{for } x \in [0, 1].$$

Hence

$$x + 3 \geq 4x^{1/4}, \quad \text{for } x \in [0, 1].$$

(c) 1. Let

$$f(x) = \log(1 + x) - (x - \tfrac{1}{2}x^2) \quad (x \in [0, \infty)).$$

Then f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$.

2. We have

$$\begin{aligned} f'(x) &= \frac{1}{1 + x} - (1 - x) \\ &= \frac{1 - (1 - x^2)}{1 + x} \\ &= \frac{x^2}{1 + x} > 0, \quad \text{for } x \in (0, \infty), \end{aligned}$$

and $f(0) = \log 1 - 0 = 0$.

It follows that f is *strictly* increasing on $[0, \infty)$, so

$$f(x) > f(0) = 0, \quad \text{for } x \in (0, \infty).$$

Hence

$$\log(1 + x) > x - \tfrac{1}{2}x^2, \quad \text{for } x \in (0, \infty).$$

Solution to Additional Exercise F23

(a) Let $I = \mathbb{R}$ and define

$$f(x) = 1 - \cos x \quad \text{and} \quad g(x) = x^2 \quad (x \in \mathbb{R}).$$

Then f and g are differentiable, and

$$f(0) = g(0) = 0.$$

Thus f and g satisfy the conditions of l'Hôpital's Rule at 0.

Now,

$$f'(x) = \sin x \quad \text{and} \quad g'(x) = 2x.$$

Thus, by l'Hôpital's Rule, the required limit exists and equals

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}.$$

(b) Let $I = (0, 2)$ say (or any other interval containing 1 and with all $x \in I$ satisfying $5x + 3 \geq 0$) and define

$$f(x) = (5x+3)^{1/3} - (x+3)^{1/2} \quad \text{and} \quad g(x) = x - 1,$$

where $x \in I$.

Then f and g are differentiable on I, and

$$f(1) = g(1) = 0.$$

Thus f and g satisfy the conditions of l'Hôpital's Rule at 1.

Now,

$$f'(x) = \tfrac{5}{3}(5x+3)^{-2/3} - \tfrac{1}{2}(x+3)^{-1/2}$$

and

$$g'(x) = 1.$$

Since $g'(1) = 1 \neq 0$, and f' and g' are both continuous, we deduce, by the Combination Rules for continuous functions, that

$$\lim_{x\to 1} \frac{f'(x)}{g'(x)} = \frac{f'(1)}{g'(1)} = \frac{\frac{5}{3} \times \frac{1}{4} - \frac{1}{2} \times \frac{1}{2}}{1} = \frac{1}{6}.$$

Thus, by l'Hôpital's Rule, the required limit exists and equals $\frac{1}{6}$.

(c) Let $I = \mathbb{R}$ and define

$$f(x) = \sinh x - x \quad \text{and} \quad g(x) = \sin(x^2) \quad (x \in \mathbb{R}).$$

Then f and g are differentiable on $\mathbb{R}$, and

$$f(0) = g(0) = 0.$$

Thus f and g satisfy the conditions of l'Hôpital's Rule at 0.

Now,

$$f'(x) = \cosh x - 1 \quad \text{and} \quad g'(x) = 2x\cos(x^2).$$

Thus, by l'Hôpital's Rule, the required limit exists and equals

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cosh x - 1}{2x\cos(x^2)}, \qquad (*1)$$

provided that limit $(*1)$ exists. Here

$$f'(0) = g'(0) = 0$$

and so we cannot apply l'Hôpital's Rule at this stage. However, both f' and g' are differentiable on $\mathbb{R}$ and $f'(0) = g'(0) = 0$ so f' and g' satisfy the conditions of l'Hôpital's Rule at 0.

Now,

$$f''(x) = \sinh x$$

and

$$g''(x) = 2\cos(x^2) - 4x^2\sin(x^2) \quad (x \in \mathbb{R}).$$

Thus, by l'Hôpital's Rule, the required limit exists and equals

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{\sinh x}{2\cos(x^2) - 4x^2\sin(x^2)}, \qquad (*2)$$

provided that limit $(*2)$ exists.

Since $g''(0) = 2 \neq 0$, and f'' and g'' are both continuous, we deduce, by the Combination Rules for continuous functions, that

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{f''(0)}{g''(0)} = \frac{0}{2} = 0.$$

Hence limit $(*2)$ exists and equals 0.

It follows that limit $(*1)$ exists and equals 0, so the required limit also exists and equals 0.

(d) Let $I = \mathbb{R}$ and define

$$f(x) = \sinh(x + \sin x) \quad \text{and} \quad g(x) = \sin x \quad (x \in \mathbb{R}).$$

Then f and g are differentiable on $\mathbb{R}$, and

$$f(0) = g(0) = 0.$$

Thus f and g satisfy the conditions of l'Hôpital's Rule at 0.

Now

$$f'(x) = (1 + \cos x)\cosh(x + \sin x)$$

and

$$g'(x) = \cos x.$$

Since $g'(0) = 1 \neq 0$, and f' and g' are continuous, we deduce, by the Combination Rules for continuous functions, that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{f'(0)}{g'(0)} = \frac{2\cosh 0}{1} = 2.$$

Thus, by l'Hôpital's Rule, the required limit exists and equals 2.

Additional exercises for Unit F3

Section 1

Additional Exercise F24

Consider the function

$$f(x) = \begin{cases} 1-|x|, & -1 < x < 1, \\ 1, & x = \pm 1. \end{cases}$$

Sketch the graph of f and identify $\min f$, $\max f$, $\inf f$ and $\sup f$ (if they exist).

Additional Exercise F25

Let f be the function

$$f(x) = \begin{cases} |x|, & -1 < x < 1, \\ \frac{1}{2}, & x = \pm 1. \end{cases}$$

Evaluate $L(f,P)$ and $U(f,P)$ for each of the following partitions P of $[-1,1]$.

(a) $P = \{[-1,-\frac{1}{2}],[-\frac{1}{2},0],[0,\frac{1}{2}],[\frac{1}{2},1]\}$

(b) $P = \{[-1,-\frac{1}{4}],[-\frac{1}{4},\frac{1}{3}],[\frac{1}{3},1]\}$

Additional Exercise F26

Let f be the function

$$f(x) = \begin{cases} 1-x, & 0 \le x < 1, \\ 2, & x = 1. \end{cases}$$

(a) Using the standard partition P_n of $[0,1]$, evaluate $L(f,P_n)$ and $U(f,P_n)$.

(b) Deduce that f is integrable on $[0,1]$, and evaluate $\displaystyle\int_0^1 f$.

Additional Exercise F27

Use Theorem F45 to prove that the constant function

$$f(x) = c \quad (x \in [a,b])$$

is integrable on $[a,b]$ and

$$\int_a^b f = (b-a)c.$$

Additional Exercise F28

Prove that the function

$$f(x) = \begin{cases} 1+x, & 0 \le x \le 1,\ x \text{ rational}, \\ 1-x, & 0 \le x \le 1,\ x \text{ irrational}, \end{cases}$$

is not integrable on $[0,1]$.

Additional Exercise F29

Prove that if the functions f and g are integrable on $[a,b]$, then so is the function $\max\{f,g\}$.

Hint:
$\max\{f(x),g(x)\} = \frac{1}{2}\big(f(x)+g(x)+|f(x)-g(x)|\big)$.

Section 2

Additional Exercise F30

Using the table of standard primitives and the Combination Rules, write down a primitive $F(x)$ of each of the following functions.

(a) $f(x) = (x^2-9)^{1/2} \quad (x \in (3,\infty))$

(b) $f(x) = \sin(2x+3) - 4\cos(3x-2) \quad (x \in \mathbb{R})$

(c) $f(x) = e^{2x}\sin 3x \quad (x \in \mathbb{R})$

Additional Exercise F31

For the function f in Additional Exercise F30(c), find a primitive F for which $F(\pi) = 0$.

Additional Exercise F32

Evaluate each of the following integrals, using the suggested substitution where given.

(a) $\displaystyle\int_0^{\pi/2} \tan(\sin x)\cos x\,dx, \quad u = \sin x.$

(b) $\displaystyle\int_0^1 \frac{(\tan^{-1} x)^2}{1+x^2}\,dx, \quad u = \tan^{-1} x.$

(c) $\displaystyle\int_0^{\pi/2} \frac{\sin 2x}{1+3\cos^2 x}\,dx$

(d) $\displaystyle\int_1^e 8x^7 \log x\,dx$

(e) $\displaystyle\int_e^{e^2} \frac{\log(\log x)}{x}\,dx$

(f) $\displaystyle\int_1^4 \frac{dx}{(1+x)\sqrt{x}}, \quad u=\sqrt{x}.$

(g) $\displaystyle\int_0^{\pi/2} \frac{dx}{2+\cos x}, \quad u=\tan(\tfrac{1}{2}x).$

Hint: In part (g), use the identity

$$\cos x = \frac{\cos^2(\frac{1}{2}x)-\sin^2(\frac{1}{2}x)}{\cos^2(\frac{1}{2}x)+\sin^2(\frac{1}{2}x)} = \frac{1-\tan^2(\frac{1}{2}x)}{1+\tan^2(\frac{1}{2}x)}.$$

Additional Exercise F33

Let $I_n = \displaystyle\int_1^e x(\log x)^n\,dx, \quad n=0,1,2,\ldots.$

(a) Prove that

$$I_n = \tfrac{1}{2}e^2 - \tfrac{1}{2}nI_{n-1}, \quad \text{for } n=1,2,\ldots.$$

(b) Evaluate I_0, I_1, I_2 and I_3.

Section 3

Additional Exercise F34

Prove the following inequalities.

(a) $\displaystyle\int_0^1 x^3\sqrt{2(1+x^{99})}\,dx \le \frac{1}{2}$

(b) $\displaystyle\int_0^1 \frac{x^4}{(1+3x^{97})^{1/2}}\,dx \ge \frac{1}{10}$

Additional Exercise F35

Prove the following inequalities.

(a) $\displaystyle\frac{1}{2} \le \int_0^1 \frac{1+x^{30}}{2-x^{99}}\,dx \le 2$

(b) $\displaystyle\left|\int_0^2 \frac{x^2(x-3)\sin 9x}{1+x^{20}}\,dx\right| \le 4$

Additional Exercise F36

(a) Show that

$$\int \frac{dx}{x(\log x)^{3/2}} = -2(\log x)^{-1/2},$$

and hence prove that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{3/2}} \text{ is convergent.}$$

(b) Show that

$$\int \frac{dx}{x\log x} = \log(\log x),$$

and hence prove that

$$\sum_{n=2}^{\infty} \frac{1}{n\log n} \text{ is divergent.}$$

Section 4

Additional Exercise F37

Use Stirling's Formula to estimate each of the following numbers (giving your answers to two significant figures).

(a) $\displaystyle\binom{400}{200}\left(\frac{1}{2}\right)^{400}$ (b) $\displaystyle\frac{400!\sqrt{800\pi}}{(100!)^4\,4^{400}}$

Additional Exercise F38

Use Stirling's Formula to determine a number λ such that

$$\frac{(8n)!}{((2n)!)^4} \sim \lambda\frac{4^{8n-1}}{n^{3/2}} \quad \text{as } n\to\infty.$$

Additional Exercise F39

Use Stirling's Formula to prove that

$$\binom{3n}{n} \sim \sqrt{\frac{3}{4\pi n}}\,\frac{3^{3n}}{2^{2n}} \quad \text{as } n\to\infty.$$

Additional Exercise F40

Let A be the set of positive functions with domain $\mathbb{N}$ and consider the relation defined on A by

$$f\sim g \quad \text{if} \quad \frac{f(n)}{g(n)} \to 1 \quad \text{as } n\to\infty.$$

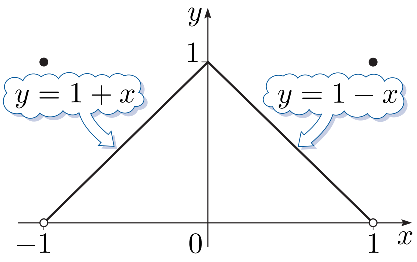
Show that this relation is an equivalence relation.

Remark: You met this relation in connection with Stirling's Formula in Section 4 of Unit F3. There, we wrote $f(n)\sim g(n)$ as $n\to\infty$ or simply $f(n)\sim g(n)$ as shorthand for the definition given above.

Solutions to additional exercises for Unit F3

Solution to Additional Exercise F24

The graph of f is shown below.



First, $\inf f = 0$ since

1. $f(x) \geq 0$, for all $x \in [-1, 1]$,
2. if $m' > 0$, then m' is not a lower bound for f on $[-1, 1]$ because the sequence $(1 - 1/n)$ is contained in $[-1, 1]$ and
$$\begin{aligned} f(1 - 1/n) &= 1 - (1 - 1/n) \\ &= \frac{1}{n} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Also, $\min f$ does not exist, since there is no point x such that $f(x) = 0$.

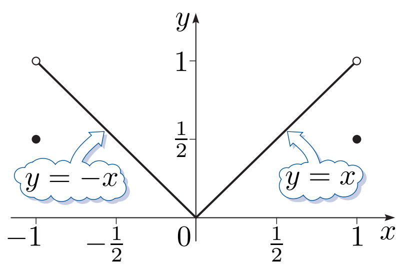
Next, $\max f = 1$ since

1. $f(x) \leq 1$, for all $x \in [-1, 1]$,
2. $f(-1) = f(0) = f(1) = 1$.

Finally, $\sup f = 1$, since f has maximum 1 on $[-1, 1]$.

Solution to Additional Exercise F25

The graph of f is shown below. (This is included to aid your understanding – you do not need to sketch the graph as part of your solution.)



(a)

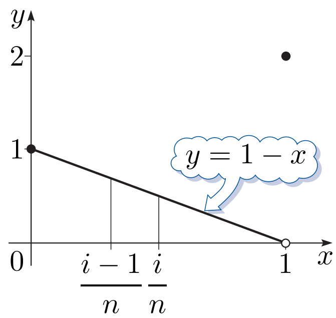
$$\begin{aligned} L(f, P) &= \sum_{i=1}^{4} m_i \, \delta x_i \\ &= \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(0 \times \tfrac{1}{2}\right) + \left(0 \times \tfrac{1}{2}\right) + \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) \\ &= \tfrac{1}{2} \\ U(f, P) &= \sum_{i=1}^{4} M_i \, \delta x_i \\ &= \left(1 \times \tfrac{1}{2}\right) + \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(1 \times \tfrac{1}{2}\right) \\ &= \tfrac{3}{2} \end{aligned}$$

(b)

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{3} m_i \, \delta x_i \\ &= \left(\tfrac{1}{4} \times \tfrac{3}{4}\right) + \left(0 \times \tfrac{7}{12}\right) + \left(\tfrac{1}{3} \times \tfrac{2}{3}\right) = \tfrac{59}{144} \\ U(f, P) &= \sum_{i=1}^{3} M_i \, \delta x_i \\ &= \left(1 \times \tfrac{3}{4}\right) + \left(\tfrac{1}{3} \times \tfrac{7}{12}\right) + \left(1 \times \tfrac{2}{3}\right) = \tfrac{29}{18} \end{aligned}$$

Solution to Additional Exercise F26

The graph of f is shown below. (This is included to aid your understanding – you do not need to sketch the graph as part of your solution.)



(a) We have

$$P_n = \left\{ \left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right] \right\}.$$

On $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, for $i = 1, 2, \ldots, n - 1$, we have

$$\begin{aligned} m_i &= f\left(\frac{i}{n}\right) = 1 - \frac{i}{n}, \\ M_i &= f\left(\frac{i-1}{n}\right) = 1 - \frac{i-1}{n}. \end{aligned}$$

Also, $m_n = 0$, $M_n = f(1) = 2$, and

$$\delta x_i = \frac{1}{n}, \quad \text{for } i = 1, 2, \ldots, n.$$

Hence

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} m_i \, \delta x_i \\ &= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{1}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{i}{n}\right) \\ &= \frac{1}{n} \times n - \frac{1}{n^2}\sum_{i=1}^{n} i \\ &= 1 - \frac{1}{n^2} \times \frac{n(n+1)}{2} \\ &= \frac{1}{2} - \frac{1}{2n} \to \frac{1}{2} \quad \text{as } n \to \infty, \end{aligned}$$

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} M_i \, \delta x_i \\ &= \sum_{i=1}^{n-1} f\left(\frac{i-1}{n}\right)\frac{1}{n} + \left(2 \times \frac{1}{n}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n-1}\left(1 - \frac{i-1}{n}\right) + \frac{2}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n-1}\left(1 + \frac{1}{n} - \frac{i}{n}\right) + \frac{2}{n} \\ &= \frac{n-1}{n}\left(1 + \frac{1}{n}\right) - \frac{1}{n^2}\sum_{i=1}^{n-1} i + \frac{2}{n} \\ &= \frac{n^2-1}{n^2} - \frac{1}{n^2} \times \frac{(n-1)n}{2} + \frac{2}{n} \\ &= \frac{1}{2} - \frac{1}{n^2} + \frac{5}{2n} \to \frac{1}{2} \quad \text{as } n \to \infty. \end{aligned}$$

(b) Since $\|P_n\| \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \tfrac{1}{2},$$

it follows from Theorem F45 that f is integrable on $[0, 1]$ and

$$\int_0^1 f = \tfrac{1}{2}.$$

Solution to Additional Exercise F27

The standard partition P_n of $[a, b]$ consists of n subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, each of length $\delta x_i = (b - a)/n$. Also,

$$m_i = M_i = c, \quad \text{for } i = 1, 2, \ldots, n.$$

Thus

$$L(f, P_n) = \sum_{i=1}^{n} m_i \, \delta x_i = \sum_{i=1}^{n} c \, \frac{b-a}{n} = c(b-a),$$

$$U(f, P_n) = \sum_{i=1}^{n} M_i \, \delta x_i = \sum_{i=1}^{n} c \, \frac{b-a}{n} = c(b-a).$$

Since $\|P_n\| \to 0$ as $n \to \infty$, and

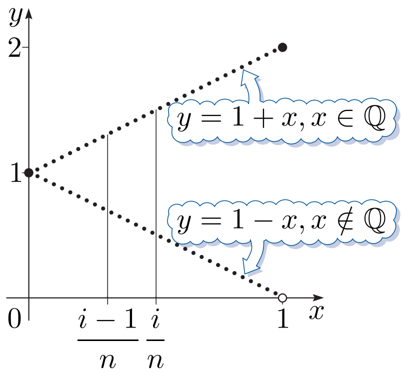
$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = c(b-a),$$

it follows from Theorem F45 that f is integrable on $[a, b]$ and

$$\int_a^b f = c(b-a).$$

Solution to Additional Exercise F28

The graph of f is shown below. (This is included to aid your understanding – you do not need to sketch the graph as part of your solution.)



Let P_n be the standard partition of $[0, 1]$:

$$\left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right]\right\}.$$

Both rational and irrational points are dense in the subinterval

$$\left[\frac{i-1}{n}, \frac{i}{n}\right], \quad \text{for } i = 1, 2, \ldots, n.$$

Also,

at the rational points, $f(x) = 1 + x \geq 1$,
at the irrational points, $f(x) = 1 - x \leq 1$.

Hence

$$m_i = 1 - \frac{i}{n} \quad \text{and} \quad M_i = 1 + \frac{i}{n}.$$

Also,

$$\delta x_i = \frac{1}{n}, \quad \text{for } i = 1, 2, \ldots, n.$$

Hence

$$\begin{aligned}
L(f,P_n) &= \sum_{i=1}^{n} m_i\,\delta x_i\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{i}{n}\right)\\
&= \frac{1}{n}\times n - \frac{1}{n^2}\sum_{i=1}^{n} i\\
&= 1-\frac{1}{n^2}\times\frac{n(n+1)}{2}\\
&= \frac{1}{2}-\frac{1}{2n},\\
U(f,P_n) &= \sum_{i=1}^{n} M_i\,\delta x_i\\
&= \frac{1}{n}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\\
&= \frac{1}{n}\times n + \frac{1}{n^2}\sum_{i=1}^{n} i\\
&= 1+\frac{1}{n^2}\times\frac{n(n+1)}{2}\\
&= \frac{3}{2}+\frac{1}{2n}.
\end{aligned}$$

Now $\|P_n\| \to 0$ as $n\to\infty$, and

$$\lim_{n\to\infty} L(f,P_n) = \tfrac{1}{2} \neq \tfrac{3}{2} = \lim_{n\to\infty} U(f,P_n).$$

Thus f is not integrable on $[0,1]$, by Theorem F46.

Solution to Additional Exercise F29

Since f and g are integrable on $[a,b]$, so is $f-g$, by the Sum and Multiple Rules, and so therefore is $|f-g|$, by the Modulus Rule.

Hence the function

$$\max\{f,g\} = \tfrac{1}{2}\left(f+g+|f-g|\right)$$

is integrable on $[a,b]$, by the Sum and Multiple Rules.

Solution to Additional Exercise F30

Using the table of standard primitives and the Combination Rules, we obtain the following primitives.

(a) $F(x) = \frac{1}{2}x(x^2-9)^{1/2} - \frac{9}{2}\log(x+(x^2-9)^{1/2})$

(b) $F(x) = -\frac{1}{2}\cos(2x+3) - \frac{4}{3}\sin(3x-2)$

(c) $F(x) = \frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x)$

Solution to Additional Exercise F31

Since all primitives of f differ only by a constant, by Theorem F55, we take

$$F(x) = c + \tfrac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x)$$

and choose a value of c such that $F(\pi)=0$. Thus

$$0 = F(\pi) = c + \tfrac{1}{13}e^{2\pi}(2\sin 3\pi - 3\cos 3\pi),$$

so

$$c = -\tfrac{3}{13}e^{2\pi}.$$

Solution to Additional Exercise F32

(a) Let $u=\sin x$; then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x\,dx.$$

Also,

$$\begin{aligned}
&\text{when } x=0, && u=0,\\
&\text{when } x=\pi/2, && u=1.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^{\pi/2} \tan(\sin x)\cos x\,dx &= \int_0^1 \tan u\,du\\
&= [\log(\sec u)]_0^1\\
&= \log(\sec 1) - \log 1\\
&= \log(\sec 1).
\end{aligned}$$

(b) Let $u=\tan^{-1}x$, so $x=\tan u$; then

$$\frac{dx}{du} = \sec^2 u, \quad \text{so} \quad dx = \sec^2 u\,du.$$

Also,

$$\begin{aligned}
&\text{when } x=0, && u=0,\\
&\text{when } x=1, && u=\pi/4.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}\,dx\\
&= \int_0^{\pi/4} \frac{u^2\sec^2 u}{1+\tan^2 u}\,du\\
&= \int_0^{\pi/4} u^2\,du \quad (\text{since } 1+\tan^2 u = \sec^2 u)\\
&= \left[\tfrac{1}{3}u^3\right]_0^{\pi/4} = \pi^3/192.
\end{aligned}$$

(c) Since

$$\frac{\sin 2x}{1+3\cos^2 x} = \frac{2\sin x\cos x}{1+3\cos^2 x}$$

and

$$g'(x) = -6\cos x\sin x, \quad \text{where } g(x) = 1+3\cos^2 x,$$

the integral is of the form $-\frac{1}{3}\int g'(x)/g(x)\,dx$. Thus, by equation (17) in Section 2 of Unit F3,

$$\begin{aligned}\int_0^{\pi/2}\frac{\sin 2x}{1+3\cos^2 x}\,dx &= \left[-\tfrac{1}{3}\log(1+3\cos^2 x)\right]_0^{\pi/2}\\ &= -\tfrac{1}{3}\log 1+\tfrac{1}{3}\log 4\\ &= \tfrac{2}{3}\log 2.\end{aligned}$$

(d) Here we use integration by parts, with

$$f(x)=\log x \quad\text{and}\quad g'(x)=8x^7;$$

then

$$f'(x)=1/x \quad\text{and}\quad g(x)=x^8.$$

Hence

$$\begin{aligned}\int_1^e 8x^7\log x\,dx &= \left[x^8\log x\right]_1^e-\int_1^e x^8\frac{1}{x}\,dx\\ &= e^8-\int_1^e x^7\,dx\\ &= e^8-\left[\tfrac{1}{8}x^8\right]_1^e\\ &= e^8-\tfrac{1}{8}e^8+\tfrac{1}{8}=\tfrac{1}{8}(7e^8+1).\end{aligned}$$

(e) Here we use integration by parts, with

$$f(x)=\log(\log x) \quad\text{and}\quad g'(x)=\frac{1}{x};$$

then

$$f'(x)=\frac{1}{\log x}\times\frac{1}{x} \quad\text{and}\quad g(x)=\log x.$$

Hence

$$\begin{aligned}&\int_e^{e^2}\frac{\log(\log x)}{x}\,dx\\ &= \left[\log(\log x)\log x\right]_e^{e^2}-\int_e^{e^2}\frac{\log x}{x\log x}\,dx\\ &= 2\log 2-\log 1-\left[\log x\right]_e^{e^2}\\ &= 2\log 2-(2-1)=2\log 2-1.\end{aligned}$$

(f) Let $u=\sqrt{x}$, so $x=u^2$; then

$$\frac{dx}{du}=2u, \quad\text{so}\quad dx=2u\,du.$$

Also,

when $x=1$, $u=1$,
when $x=4$, $u=2$.

Hence

$$\begin{aligned}\int_1^4\frac{dx}{(1+x)\sqrt{x}} &= \int_1^2\frac{2u}{(1+u^2)u}\,du\\ &= \left[2\tan^{-1}u\right]_1^2\\ &= 2\tan^{-1}2-\pi/2.\end{aligned}$$

(g) Let $u=\tan(\frac{1}{2}x)$; then

$$\frac{du}{dx}=\tfrac{1}{2}\sec^2(\tfrac{1}{2}x), \quad\text{so}\quad dx=\frac{2}{1+u^2}\,du,$$

since $\sec^2(\frac{1}{2}x)=1+\tan^2(\frac{1}{2}x)$.

Also,

when $x=0$, $u=0$,
when $x=\pi/2$, $u=1$.

Next, by the given identity,

$$\cos x=\frac{1-\tan^2(\frac{1}{2}x)}{1+\tan^2(\frac{1}{2}x)}=\frac{1-u^2}{1+u^2},$$

so

$$\frac{1}{2+\cos x}=\frac{1}{2+(1-u^2)/(1+u^2)}=\frac{1+u^2}{3+u^2}.$$

Hence

$$\begin{aligned}\int_0^{\pi/2}\frac{dx}{2+\cos x} &= \int_0^1\frac{1+u^2}{3+u^2}\,\frac{2}{1+u^2}\,du\\ &= 2\int_0^1\frac{du}{3+u^2}\\ &= \frac{2}{\sqrt{3}}\left[\tan^{-1}(u/\sqrt{3})\right]_0^1\\ &= \frac{2}{\sqrt{3}}\times\frac{\pi}{6}=\frac{\sqrt{3}\,\pi}{9}.\end{aligned}$$

Solution to Additional Exercise F33

(a) Here we use integration by parts, with

$$f(x)=(\log x)^n \quad\text{and}\quad g'(x)=x;$$

then

$$f'(x)=n(\log x)^{n-1}x^{-1} \quad\text{and}\quad g(x)=\tfrac{1}{2}x^2.$$

Hence

$$\begin{aligned}I_n &= \left[\tfrac{1}{2}x^2(\log x)^n\right]_1^e-\int_1^e\tfrac{1}{2}x^2 n(\log x)^{n-1}x^{-1}\,dx\\ &= \tfrac{1}{2}e^2-\tfrac{1}{2}n\int_1^e x(\log x)^{n-1}\,dx\\ &= \tfrac{1}{2}e^2-\tfrac{1}{2}nI_{n-1}, \quad\text{for } n\geq 1.\end{aligned}$$

(b) First,

$$\begin{aligned}I_0 &= \int_1^e x\,dx\\ &= \left[\tfrac{1}{2}x^2\right]_1^e=\tfrac{1}{2}e^2-\tfrac{1}{2}.\end{aligned}$$

Using the formula from part (a), we obtain

$$\begin{aligned}I_1 &= \tfrac{1}{2}e^2-\tfrac{1}{2}I_0\\ &= \tfrac{1}{2}e^2-\tfrac{1}{2}\left(\tfrac{1}{2}e^2-\tfrac{1}{2}\right)=\tfrac{1}{4}e^2+\tfrac{1}{4},\\ I_2 &= \tfrac{1}{2}e^2-\tfrac{1}{2}\times 2\times I_1\\ &= \tfrac{1}{2}e^2-\left(\tfrac{1}{4}e^2+\tfrac{1}{4}\right)=\tfrac{1}{4}e^2-\tfrac{1}{4},\\ I_3 &= \tfrac{1}{2}e^2-\tfrac{1}{2}\times 3\times I_2\\ &= \tfrac{1}{2}e^2-\tfrac{3}{2}\left(\tfrac{1}{4}e^2-\tfrac{1}{4}\right)=\tfrac{1}{8}e^2+\tfrac{3}{8}.\end{aligned}$$

Solution to Additional Exercise F34

(a) If $x \in [0,1]$, then

$$x^3\sqrt{2(1+x^{99})} \le x^3\sqrt{2(1+1)} = 2x^3.$$

Hence, by Inequality Rule (a),

$$\int_0^1 x^3\sqrt{2(1+x^{99})}\,dx \le \int_0^1 2x^3\,dx = \left[\tfrac{1}{2}x^4\right]_0^1 = \tfrac{1}{2}.$$

(b) If $x \in [0,1]$, then

$$1+3x^{97} \le 1+3 = 4,$$

so

$$\frac{x^4}{(1+3x^{97})^{1/2}} \ge \frac{x^4}{4^{1/2}} = \tfrac{1}{2}x^4.$$

Hence, by Inequality Rule (a),

$$\int_0^1 \frac{x^4}{(1+3x^{97})^{1/2}}\,dx \ge \int_0^1 \tfrac{1}{2}x^4\,dx = \left[\tfrac{1}{10}x^5\right]_0^1 = \tfrac{1}{10}.$$

Solution to Additional Exercise F35

(a) If $x \in [0,1]$, then

$$1 \le 1+x^{30} \le 1+1 = 2$$

and

$$1 \le 2-x^{99} \le 2,$$

so

$$\tfrac{1}{2} \le \frac{1}{2-x^{99}} \le 1.$$

Hence

$$\tfrac{1}{2} \le \frac{1+x^{30}}{2-x^{99}} \le 2, \quad \text{for } x \in [0,1].$$

Since the length of $[0,1]$ is 1, it follows, by Inequality Rule (b), that

$$\tfrac{1}{2} \le \int_0^1 \frac{1+x^{30}}{2-x^{99}}\,dx \le 2.$$

(b) If $x \in [0,2]$, then $1+x^{20} \ge 1$ and $|\sin 9x| \le 1$, so

$$\begin{aligned}\left|\frac{x^2(x-3)\sin 9x}{1+x^{20}}\right| &\le x^2|x-3| \\ &= x^2(3-x) \\ &= 3x^2-x^3.\end{aligned}$$

Hence, by the Triangle Inequality and Inequality Rule (a),

$$\begin{aligned}\left|\int_0^2 \frac{x^2(x-3)\sin 9x}{1+x^{20}}\,dx\right| &\le \int_0^2 \left|\frac{x^2(x-3)\sin 9x}{1+x^{20}}\right| dx \\ &\le \int_0^2 (3x^2-x^3)\,dx \\ &= \left[x^3-\tfrac{1}{4}x^4\right]_0^2 \\ &= 8-4 = 4.\end{aligned}$$

Solution to Additional Exercise F36

(a) Let $u = \log x$; then

$$\frac{du}{dx} = \frac{1}{x}, \quad \text{so} \quad du = \frac{dx}{x}.$$

Hence

$$\int \frac{dx}{x(\log x)^{3/2}} = \int \frac{du}{u^{3/2}} = -\frac{2}{u^{1/2}} = -\frac{2}{(\log x)^{1/2}}.$$

Now let

$$f(x) = \frac{1}{x(\log x)^{3/2}} \quad (x \in [2,\infty)).$$

Then f is positive and decreasing on $[2,\infty)$, and

$$f(x) \to 0 \ \text{ as } x \to \infty.$$

Also, for $n \ge 2$,

$$\begin{aligned}\int_2^n f &= \int_2^n \frac{dx}{x(\log x)^{3/2}} \\ &= \left[\frac{-2}{(\log x)^{1/2}}\right]_2^n \\ &= 2\left(\frac{1}{(\log 2)^{1/2}} - \frac{1}{(\log n)^{1/2}}\right) \le \frac{2}{(\log 2)^{1/2}}.\end{aligned}$$

Since the sequence $\left(\int_2^n f\right)_2^\infty$ is bounded above, it follows from part (a) of the Integral Test that the series converges.

(b) Let $u = \log x$, as in part (a). Then

$$\int \frac{dx}{x\log x} = \int \frac{du}{u} = \log u = \log(\log x).$$

Now let

$$f(x) = \frac{1}{x\log x} \quad (x \in [2,\infty)).$$

Then f is positive and decreasing on $[2,\infty)$, and

$$f(x) \to 0 \ \text{ as } x \to \infty.$$

Also, for $n \geq 2$,

$$\begin{aligned}\int_2^n f &= \int_2^n \frac{dx}{x\log x}\\ &= \left[\log(\log x)\right]_2^n\\ &= \log(\log n) - \log(\log 2) \to \infty \quad \text{as } n \to \infty.\end{aligned}$$

Hence, by part (b) of the Integral Test, the series diverges.

Solution to Additional Exercise F37

In each part we approximate the factorials using Stirling's Formula.

(a)

$$\begin{aligned}\binom{400}{200}\left(\frac{1}{2}\right)^{400} &= \frac{400!}{200!\,200!\,2^{400}}\\ &\approx \frac{\sqrt{800\pi}\,(400/e)^{400}}{\left(\sqrt{400\pi}\,(200/e)^{200}\right)^2 2^{400}}\\ &= \frac{\sqrt{800\pi}}{400\pi} = \frac{\sqrt{8}}{40\sqrt{\pi}}\\ &= \frac{\sqrt{2}}{20\sqrt{\pi}} = 0.040 \quad \text{(to 2 s.f.)}.\end{aligned}$$

(b)

$$\begin{aligned}\frac{400!\,\sqrt{800\pi}}{(100!)^4\,4^{400}} &\approx \frac{\sqrt{800\pi}\,(400/e)^{400}\,\sqrt{800\pi}}{\left(\sqrt{200\pi}\,(100/e)^{100}\right)^4\,4^{400}}\\ &= \frac{800\pi}{(\sqrt{200\pi}\,)^4}\\ &= \frac{1}{50\pi} = 0.0064 \quad \text{(to 2 s.f.)}.\end{aligned}$$

Solution to Additional Exercise F38

By Stirling's Formula, and the Product and Quotient Rules for $\sim$, we obtain

$$\begin{aligned}\frac{(8n)!}{((2n)!)^4} &\sim \frac{\sqrt{16\pi n}\,(8n/e)^{8n}}{\left(\sqrt{4\pi n}\,(2n/e)^{2n}\right)^4}\\ &= \frac{\sqrt{16\pi n}\,4^{8n}}{(\sqrt{4\pi n})^4} = \frac{1}{\pi^{3/2}}\,\frac{4^{8n-1}}{n^{3/2}}.\end{aligned}$$

Hence

$$\lambda = 1/\pi^{3/2}.$$

Solution to Additional Exercise F39

By Stirling's Formula, and the Product and Quotient Rules for $\sim$, we obtain

$$\begin{aligned}\binom{3n}{n} &= \frac{(3n)!}{n!\,(2n)!}\\ &\sim \frac{\sqrt{6\pi n}\,(3n/e)^{3n}}{\sqrt{2\pi n}\,(n/e)^n\,\sqrt{4\pi n}\,(2n/e)^{2n}}\\ &= \frac{\sqrt{3}}{\sqrt{4\pi n}}\,\frac{(3n)^{3n}}{n^n(2n)^{2n}}\\ &= \sqrt{\frac{3}{4\pi n}}\,\frac{3^{3n}}{2^{2n}},\end{aligned}$$

as required.

Solution to Additional Exercise F40

Recall from Unit A3, that we need to show that the relation is reflexive, symmetric and transitive. As earlier, we label these three properties as E1, E2 and E3, respectively.

E1 Let $f \in A$. Then

$$\frac{f(n)}{f(n)} = 1 \to 1 \quad \text{as } n \to \infty.$$

Thus $\sim$ is reflexive.

E2 Let $f, g \in A$ and suppose that $f \sim g$. Then

$$\frac{f(n)}{g(n)} \to 1 \quad \text{as } n \to \infty.$$

It follows from the Quotient Rule for limits that

$$\frac{g(n)}{f(n)} = \frac{1}{f(n)/g(n)} \to \frac{1}{1} = 1 \quad \text{as } n \to \infty.$$

Hence $g \sim f$ and so $\sim$ is symmetric.

E3 Let $f, g, h \in A$ and suppose that $f \sim g$ and $g \sim h$. Then

$$\frac{f(n)}{g(n)} \to 1 \quad \text{and} \quad \frac{g(n)}{h(n)} \to 1 \quad \text{as } n \to \infty.$$

It follows from the Product Rule for limits that

$$\frac{f(n)}{h(n)} = \frac{f(n)}{g(n)} \times \frac{g(n)}{h(n)} \to \frac{1}{1} = 1 \quad \text{as } n \to \infty.$$

Hence $f \sim h$ and so $\sim$ is transitive.

We have now shown that $\sim$ is an equivalence relation.

Additional exercises for Unit F4

Section 1

Additional Exercise F41

Determine the tangent approximation to the function

$$f(x) = 2 - 3x + x^2 + e^x$$

at each of the following points a.

(a) $a = 0$ (b) $a = 1$

Additional Exercise F42

Determine the Taylor polynomial of degree 3 for each of the following functions f at the given point a.

(a) $f(x) = \log(1 + x), \quad a = 2.$

(b) $f(x) = \sin x, \quad a = \pi/6.$

(c) $f(x) = (1 + x)^{-2}, \quad a = \frac{1}{2}.$

(d) $f(x) = \tan x, \quad a = \pi/4.$

Additional Exercise F43

Determine the Taylor polynomial of degree 4 for each of the following functions f at the given point a.

(a) $f(x) = \cosh x, \quad a = 0.$

(b) $f(x) = x^5, \quad a = 1.$

Additional Exercise F44

Let T_3 be the Taylor polynomial of degree 3 at 0 for $f(x) = e^x$. Use a calculator to show that

$$|e^{0.1} - T_3(0.1)| < 5 \times 10^{-6}.$$

(The Taylor polynomial of degree n at 0 for e^x was found in Exercise F59(a) of Unit F4.)

Section 2

Additional Exercise F45

By applying Taylor's Theorem with $n = 2$ to the function $f(x) = \sin x$ at $a = \pi/4$, prove that, for $x \neq \pi/4$,

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 - \frac{\cos c}{6}\left(x - \frac{\pi}{4}\right)^3,$$

where c lies between $\pi/4$ and x.

Additional Exercise F46

Find the Taylor polynomial $T_4(x)$ at 0 for the function $f(x) = \sinh x$, and use your answer to calculate $\sinh(0.2)$ to four decimal places.

Additional Exercise F47

(a) Find the Taylor polynomial $T_3(x)$ at 2 for the function $f(x) = x/(x + 3)$.

(b) Show that $T_3(x)$ approximates $f(x)$ to within 6×10^{-5} on the interval $\left[2, \frac{5}{2}\right]$.

Section 3

Additional Exercise F48

Determine the radius of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(3n)!}{(n!)^2}(x + 2)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}(x - 5)^n$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n + 1)^n}$

Hint: In part (b) you may find it helpful to use the fact that $(1 + 1/n)^n \to e$ as $n \to \infty$.

Additional Exercise F49

Determine the interval of convergence of each of the following power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n} (x+1)^n$

Section 4

Additional Exercise F50

Determine the Taylor series at 0 for each of the following functions. In each case, indicate the general term and state a range of validity for the series.

(a) $f(x) = \log\left(\dfrac{1+2x}{1-2x}\right)$ (b) $f(x) = \dfrac{\cosh x}{1-x}$

Additional Exercise F51

Determine the first three non-zero terms in the Taylor series at 0 for the function $f(x) = e^x \sin x$, and state a range of validity for the series.

Additional Exercise F52

(a) Find the Taylor series at 0 for the function $f(x) = (1-x)^{-1/2}$.

(b) Use your solution to part (a) to find the Taylor series at 0 for the function $g(x) = \sin^{-1} x$, and state a range of validity for this series.

Solutions to additional exercises for Unit F4

Solution to Additional Exercise F41

The tangent approximation to f at a is

$$f(x) = f(a) + f'(a)(x-a).$$

(a) We have

$$\begin{aligned} &f(x) = 2 - 3x + x^2 + e^x, && f(0) = 3;\\ &f'(x) = -3 + 2x + e^x, && f'(0) = -2. \end{aligned}$$

Hence the tangent approximation at 0 to f is

$$f(x) \approx 3 - 2(x-0) = 3 - 2x.$$

(b) We have

$$\begin{aligned} &f(x) = 2 - 3x + x^2 + e^x, && f(1) = e;\\ &f'(x) = -3 + 2x + e^x, && f'(1) = -1 + e. \end{aligned}$$

Hence the tangent approximation at 1 to f is

$$f(x) \approx e + (e-1)(x-1).$$

Solution to Additional Exercise F42

(a) We have

$$\begin{aligned} &f(x) = \log(1+x) && f(2) = \log 3;\\ &f'(x) = 1/(1+x), && f'(2) = 1/3;\\ &f''(x) = -1/(1+x)^2 && f''(2) = -1/9;\\ &f^{(3)}(x) = 2/(1+x)^3, && f^{(3)}(2) = 2/27. \end{aligned}$$

Hence

$$\begin{aligned} T_3(x) = \log 3 &+ \tfrac{1}{3}(x-2) - \tfrac{1}{18}(x-2)^2\\ &+ \tfrac{1}{81}(x-2)^3. \end{aligned}$$

(b) We have

$$\begin{aligned} &f(x) = \sin x, && f(\pi/6) = 1/2;\\ &f'(x) = \cos x, && f'(\pi/6) = \sqrt{3}/2;\\ &f''(x) = -\sin x, && f''(\pi/6) = -1/2;\\ &f^{(3)}(x) = -\cos x, && f^{(3)}(\pi/6) = -\sqrt{3}/2. \end{aligned}$$

Hence

$$\begin{aligned} T_3(x) =& \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2\\ &- \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3. \end{aligned}$$

(c) We have

$$\begin{aligned} &f(x) = (1+x)^{-2}, && f(\tfrac{1}{2}) = 4/9;\\ &f'(x) = -2(1+x)^{-3}, && f'(\tfrac{1}{2}) = -16/27;\\ &f''(x) = 6(1+x)^{-4}, && f''(\tfrac{1}{2}) = 32/27;\\ &f^{(3)}(x) = -24(1+x)^{-5}, && f^{(3)}(\tfrac{1}{2}) = -256/81. \end{aligned}$$

Hence

$$T_3(x) = \tfrac{4}{9} - \tfrac{16}{27}(x - \tfrac{1}{2}) + \tfrac{16}{27}(x - \tfrac{1}{2})^2 - \tfrac{128}{243}(x - \tfrac{1}{2})^3.$$

(d) We have

$$\begin{aligned} &f(x) = \tan x, && f(\pi/4) = 1;\\ &f'(x) = \sec^2 x, && f'(\pi/4) = 2;\\ &f''(x) = 2\sec^2 x \tan x, && f''(\pi/4) = 4;\\ &f^{(3)}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x, && f^{(3)}(\pi/4) = 16. \end{aligned}$$

Hence

$$T_3(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3.$$

Solution to Additional Exercise F43

(a) We have

$$\begin{aligned} &f(x) = \cosh x, && f(0) = 1;\\ &f'(x) = \sinh x, && f'(0) = 0;\\ &f''(x) = \cosh x, && f''(0) = 1;\\ &f^{(3)}(x) = \sinh x, && f^{(3)}(0) = 0;\\ &f^{(4)}(x) = \cosh x, && f^{(4)}(0) = 1. \end{aligned}$$

Hence

$$T_4(x) = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4.$$

(b) We have

$$\begin{aligned} &f(x) = x^5, && f(1) = 1;\\ &f'(x) = 5x^4, && f'(1) = 5;\\ &f''(x) = 20x^3, && f''(1) = 20;\\ &f^{(3)}(x) = 60x^2, && f^{(3)}(1) = 60;\\ &f^{(4)}(x) = 120x, && f^{(4)}(1) = 120. \end{aligned}$$

Hence

$$\begin{aligned} T_4(x) =& 1 + 5(x-1) + 10(x-1)^2 + 10(x-1)^3\\ &+ 5(x-1)^4. \end{aligned}$$

Solution to Additional Exercise F44

From Exercise F59(a) of Unit F4, we have

$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!},$$

so

$$\begin{aligned} T_3(0.1) &= 1 + 0.1 + \tfrac{1}{2}(0.1)^2 + \tfrac{1}{6}(0.1)^3 \\ &= 1.105\,1\overline{6}. \end{aligned}$$

Since

$$e^{0.1} = 1.105\,170\,918\ldots,$$

we have

$$\begin{aligned} |e^{0.1} - T_3(0.1)| &= 1.105\,170\,918\ldots - 1.105\,1\overline{6} \\ &\le 1.105\,170\,919 - 1.105\,166\,666 \\ &= 0.000\,004\,253 < 5 \times 10^{-6}, \end{aligned}$$

as required.

Solution to Additional Exercise F45

We have

$$\begin{aligned} f(x) &= \sin x, & f(\pi/4) &= 1/\sqrt{2}; \\ f'(x) &= \cos x, & f'(\pi/4) &= 1/\sqrt{2}; \\ f''(x) &= -\sin x, & f''(\pi/4) &= -1/\sqrt{2}; \\ f^{(3)}(x) &= -\cos x. \end{aligned}$$

Hence, by applying Taylor's Theorem to f at $a = \pi/4$, with $n = 2$, we obtain

$$\begin{aligned} \sin x = \frac{1}{\sqrt{2}} &+ \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) \\ &- \frac{1}{2\sqrt{2}}\left(x - \frac{\pi}{4}\right)^2 + R_2(x), \end{aligned}$$

where

$$R_2(x) = \frac{f^{(3)}(c)}{3!}\left(x - \frac{\pi}{4}\right)^3 = -\frac{\cos c}{6}\left(x - \frac{\pi}{4}\right)^3,$$

for some c between $\pi/4$ and x, as required.

Solution to Additional Exercise F46

We have

$$\begin{aligned} f(x) &= \sinh x, & f(0) &= 0; \\ f'(x) &= \cosh x, & f'(0) &= 1; \\ f''(x) &= \sinh x, & f''(0) &= 0; \\ f^{(3)}(x) &= \cosh x, & f^{(3)}(0) &= 1; \\ f^{(4)}(x) &= \sinh x, & f^{(4)}(0) &= 0. \end{aligned}$$

Hence, with $a = 0$,

$$T_4(x) = x + \frac{1}{3!}x^3 = x + \frac{1}{6}x^3.$$

Now we use Strategy F11 with $a = 0$, $x = 0.2$ and $n = 4$.

1. First,

   $$f^{(5)}(x) = \cosh x.$$

2. Thus, for $c \in [0, 0.2]$, we have

   $$|f^{(5)}(c)| = \cosh c \le \cosh(0.2) < \cosh 1.$$

   Now

   $$\cosh 1 = \tfrac{1}{2}(e + e^{-1}) < \tfrac{1}{2}(3 + 1/2) = 7/4,$$

   so we can take $M = 7/4$.

3. Hence

   $$\begin{aligned} |f(0.2) - T_4(0.2)| &= |R_4(0.2)| \\ &\le \frac{M}{(4+1)!}|x - a|^{4+1} \\ &= \frac{7/4}{5!}|0.2 - 0|^5 \\ &= 4.\overline{6} \times 10^{-6} < 5 \times 10^{-6}. \end{aligned}$$

Finally,

$$T_4(0.2) = 0.2 + \tfrac{1}{6}(0.2)^3 = 0.201\overline{3}.$$

Hence

$$0.20132833\ldots < \sinh(0.2) < 0.20133833\ldots$$

so

$$\sinh(0.2) = 0.2013 \text{ (to 4 d.p.)}.$$

Solution to Additional Exercise F47

(a) We have

$$\begin{aligned} f(x) &= x/(x+3), & f(2) &= 2/5; \\ f'(x) &= 3/(x+3)^2, & f'(2) &= 3/25; \\ f''(x) &= -6/(x+3)^3, & f''(2) &= -6/125; \\ f^{(3)}(x) &= 18/(x+3)^4, & f^{(3)}(2) &= 18/625. \end{aligned}$$

Hence

$$T_3(x) = \tfrac{2}{5} + \tfrac{3}{25}(x-2) - \tfrac{3}{125}(x-2)^2 + \tfrac{3}{625}(x-2)^3.$$

(b) We use Strategy F12 with $I = \left[2, \frac{5}{2}\right]$, $a = 2$, $r = \frac{1}{2}$ and $n = 3$.

1. First,

   $$f^{(4)}(x) = -72/(x+3)^5.$$

2. Thus

   $$|f^{(4)}(c)| = \frac{72}{(c+3)^5} \le \frac{72}{5^5}, \quad \text{for } c \in \left[2, \tfrac{5}{2}\right],$$

   so we can take $M = 72/5^5$.

3. Hence
$$\begin{aligned}|R_3(x)| &\le \frac{M}{(3+1)!}r^{3+1}\\ &= \frac{1}{4!}\times\frac{72}{5^5}\left(\frac{1}{2}\right)^4\\ &= \frac{3}{5^5}\left(\frac{1}{2}\right)^4 = 6\times 10^{-5}.\end{aligned}$$
Thus $T_3(x)$ approximates $f(x)$ to within 6×10^{-5} on $\left[2,\frac{5}{2}\right]$.

Solution to Additional Exercise F48

(a) Here $a_n = (3n)!/(n!)^2$, for $n = 0, 1, 2, \dots$, so
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(3n+3)!}{((n+1)!)^2}\times\frac{(n!)^2}{(3n)!}\\ &= \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(n+1)}\\ &= \frac{(3+3/n)(3+2/n)}{(1+1/n)(1+1/n)}(3n+1).\end{aligned}$$
Thus $\left|\dfrac{a_{n+1}}{a_n}\right| \to \infty$ as $n\to\infty$.

Hence, by the Ratio Test, the radius of convergence is 0; the power series converges only when $x=-2$.

(b) Here $a_n = n!/n^n$, for $n = 1, 2, \dots$, so
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)!}{(n+1)^{n+1}}\times\frac{n^n}{n!}\\ &= \left(\frac{n}{n+1}\right)^n\\ &= 1/\left(1+1/n\right)^n \to 1/e \quad\text{as } n\to\infty.\end{aligned}$$
Hence, by the Ratio Test, the radius of convergence is e.

(c) Here $a_n = 1/(n+1)^n$, for $n = 0, 1, 2, \dots$, so
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{1}{(n+2)^{n+1}}\times\frac{(n+1)^n}{1}\\ &= \frac{(n+1)^n}{(n+2)^n}\times\frac{1}{n+2}\\ &\le \frac{1}{n+2}\to 0 \quad\text{as } n\to\infty.\end{aligned}$$
Hence, by the Ratio Test, the power series converges for all x; that is, $R=\infty$.

Solution to Additional Exercise F49

In each case, we apply Strategy F13.

(a) Here $a_n = 2^n/n!$, for $n = 0, 1, 2, \dots$.

1. Since
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2^{n+1}}{(n+1)!}\times\frac{n!}{2^n}\\ &= \frac{2}{n+1}\to 0 \quad\text{as } n\to\infty,\end{aligned}$$
we have $R=\infty$, by the Ratio Test.

Thus this power series converges for all x, so the interval of convergence is $\mathbb{R}$.

(In this case, there are no endpoints to check.)

(b) Here $a_n = 2^n/n$, for $n = 1, 2, \dots$.

1. Since
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{2^{n+1}}{n+1}\times\frac{n}{2^n}\\ &= \frac{2}{1+1/n}\to 2 \quad\text{as } n\to\infty,\end{aligned}$$
we have $R=\frac{1}{2}$, by the Ratio Test. Since $a=-1$, this power series
   - converges for $-\frac{3}{2} < x < -\frac{1}{2}$,
   - diverges for $x > -\frac{1}{2}$ and $x < -\frac{3}{2}$.
2. If $x = -\frac{1}{2}$, then the power series is
$$\sum_{n=1}^{\infty}\frac{2^n}{n}\left(-\tfrac{1}{2}+1\right)^n = \sum_{n=1}^{\infty}\frac{1}{n},$$
which is a basic divergent series.
If $x=-\frac{3}{2}$, then the power series is
$$\sum_{n=1}^{\infty}\frac{2^n}{n}\left(-\tfrac{3}{2}+1\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n},$$
which is convergent, by the Alternating Test.

Hence the interval of convergence is $\left[-\frac{3}{2},-\frac{1}{2}\right)$.

Solution to Additional Exercise F50

(a) We have
$$\log\left(\frac{1+2x}{1-2x}\right) = \log(1+2x) - \log(1-2x).$$
We know that, for $|x|<1$,
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$
Thus, for $|2x|<1$,
$$\begin{aligned}&\log(1+2x)\\ &= 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \cdots + (-1)^{n+1}\frac{2^n x^n}{n} + \cdots\end{aligned}$$
and
$$\begin{aligned}&\log(1-2x)\\ &= -2x - \frac{4x^2}{2} - \frac{8x^3}{3} - \cdots - \frac{2^n x^n}{n} - \cdots.\end{aligned}$$

Hence, by the Sum and Multiple Rules,

$$\begin{aligned}&\log\left(\frac{1+2x}{1-2x}\right)\\&=\log(1+2x)-\log(1-2x)\\&=4x+\frac{16x^3}{3}+\cdots+\frac{2^{2k+2}x^{2k+1}}{2k+1}+\cdots,\end{aligned}$$

for $|x|<1/2$.

(b) We know that, for $|x|<1$,

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots,$$

and, for $x\in\mathbb{R}$,

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{x^{2n}}{(2n)!}+\cdots.$$

Hence, by the Product Rule,

$$\begin{aligned}&\frac{\cosh x}{1-x}\\&=\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)(1+x+x^2+\cdots)\\&=1+x+\left(1+\frac{1}{2!}\right)x^2+\left(1+\frac{1}{2!}\right)x^3\\&\quad+\left(1+\frac{1}{2!}+\frac{1}{4!}\right)x^4+\left(1+\frac{1}{2!}+\frac{1}{4!}\right)x^5+\cdots,\end{aligned}$$

for $|x|<1$. Thus, for $|x|<1$,

$$\frac{\cosh x}{1-x}=\sum_{n=0}^{\infty}a_nx^n,\quad\text{where } a_n=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{(2k)!}.$$

Solution to Additional Exercise F51

We know that

$$e^x=1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\cdots,$$

for $x\in\mathbb{R}$, and

$$\sin x=x-\tfrac{1}{6}x^3+\cdots,$$

for $x\in\mathbb{R}$.

By the Product Rule,

$$\begin{aligned}f(x)&=e^x\sin x\\&=\left(1+x+\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\cdots\right)\left(x-\tfrac{1}{6}x^3+\cdots\right)\\&=x-\tfrac{1}{6}x^3+x^2+\tfrac{1}{2}x^3\\&\quad+\text{powers of } x \text{ greater than } 3\\&=x+x^2+\tfrac{1}{3}x^3+\cdots,\end{aligned}$$

for $x\in\mathbb{R}$.

Solution to Additional Exercise F52

(a) By the General Binomial Theorem,

$$(1-x)^{-1/2}=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-x)^n,\quad\text{for } |x|<1,$$

where

$$\binom{-\frac{1}{2}}{0}=1$$

and

$$\binom{-\frac{1}{2}}{n}=\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n+1)}{n!},\quad\text{for } n\in\mathbb{N}.$$

Hence

$$\begin{aligned}&(1-x)^{-1/2}\\&=1+\tfrac{1}{2}x+\tfrac{3}{8}x^2+\cdots+(-1)^n\binom{-\frac{1}{2}}{n}x^n+\cdots,\end{aligned}$$

for $|x|<1$.

(b) We know that

$$\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}.$$

By part (a), with x replaced by x^2, we have

$$\begin{aligned}&\frac{1}{\sqrt{1-x^2}}\\&=1+\tfrac{1}{2}x^2+\tfrac{3}{8}x^4+\cdots+(-1)^n\binom{-\frac{1}{2}}{n}x^{2n}+\cdots,\end{aligned}$$

for $|x|<1$. Hence, by the Integration Rule,

$$\begin{aligned}\sin^{-1}x=&x+\tfrac{1}{6}x^3+\tfrac{3}{40}x^5+\cdots\\&+\frac{(-1)^n}{2n+1}\binom{-\frac{1}{2}}{n}x^{2n+1}+\cdots,\end{aligned}$$

for $|x|<1$, since $\sin^{-1}0=0$.

Remark: The coefficients in this Taylor series can be written as

$$\frac{(-1)^n}{2n+1}\binom{-\frac{1}{2}}{n}=\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2^nn!\,(2n+1)},\quad\text{for } n\in\mathbb{N}.$$